18.152 PROBLEM SET 6

due May 6th 10am.

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

Problem 1. *Let* $u : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}$ *be a solution to the global Cauchy problem*

$$\begin{aligned} &u_{tt} = \Delta u && x \in \mathbb{R}^3, t \geq 0 \\ &u(x,0) = g(x), u_t(x,0) = h(x) && x \in \mathbb{R}^3, \end{aligned}$$

where g, h *are smooth. Suppose that there exists some large constant* R *such that* $g(x) = h(x) = 0$ *for* $|x| \geq R$*. Show that there exists some large constant* C *depending on* $R, g, \nabla g, h$ *such that*

$$|u(x,t)| \leq C/t$$

holds for $t > 0$*.*

Hint: Use the Kirchhoff's formula.

Problem 2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set, and* $0 < p < q$*. Show that there exists some constant* C *only depending on* p, q *and the volume of* Ω *such that*

$$\|f\|_{L^p(\Omega)} \leq C\|f\|_{L^q(\Omega)},$$

holds for any function $f : \Omega \to \mathbb{R}$ *such that* $|f|^p$ *and* $|f|^q$ *are integrable.*

Problem 3. *Let* $u : \mathbb{R}^6 \times [0, +\infty) \to \mathbb{R}$ *be a solution to the global Cauchy problem*

$$\begin{aligned} &u_{tt} = \Delta u && x \in \mathbb{R}^6, t \geq 0 \\ &u(x,0) = g(x), u_t(x,0) = h(x) && x \in \mathbb{R}^6, \end{aligned}$$

where g, h *are smooth. Suppose that there exists some large constant* R *such that* $g(x) = h(x) = 0$ *for* $|x| \geq R$*. Establish a bound for* $|u(x,t)|$ *in terms of* t, R, g, h *and the first or higher order derivatives of* g, h*.*

Hint: Energy estimate.

Problem 4. *We recall the sequence* $\{(w_i, \lambda_i)\}_{i=1}^{\infty}$ *of the Dirichlet Laplace eigenpairs introduced in Lecture 9 (B). Suppose* $\int_\Omega w_i w_j dx = 0$ *for* $i \neq j$.[1] *Let* $g, h \in C^\infty(\overline{\Omega})$ *satisfy* $g = h = 0$ *on* $\partial\Omega$, *and we define*

$$u^k(x,t) = \sum_{i=1}^{k} g_i \cos(\lambda_i^{\frac{1}{2}} t) w_i(x) + \lambda_i^{-\frac{1}{2}} h_i \sin(\lambda_i^{\frac{1}{2}} t) w_i(x),$$

where

$$g_i = \langle g, w_i \rangle_{L^2} = \int_\Omega g(x) w_i(x) dx, \qquad h_i = \langle h, w_i \rangle_{L^2} = \int_\Omega h(x) w_i(x) dx.$$

Notice that $u^k(x,t)$ *is a solution to the wave equation with zero Dirichlet data. Moreover,* $\lim_{k\to+\infty} \|g - u^k\|_{L^2} = \lim_{k\to+\infty} \|h - u_t^k\|_{L^2} = 0$.

(1) *Show that the following holds for each* $t \geq 0$.
$$\|\nabla u^k(\cdot, t)\|_{L^2}^2 \leq \|\nabla g\|_{L^2}^2 + \|h\|_{L^2}^2.$$
(2) *Show that*
$$\lim_{\min\{k,l\}\to+\infty} \sup_{t\geq 0} \|\nabla u^k(\cdot,t) - \nabla u^l(\cdot,t)\|_{L^2} = 0.$$
(3) *Briefly explain why the limit* $u(x,t) = \lim_{k\to+\infty} u^k(x,t)$ *is a smooth solution to the wave equation satisfying* $u = 0$ *on* $\partial\Omega$, $u(x,0) = g(x)$, *and* $u_t(x,0) = h(x)$.

[1]See lecture 10 (A) where we discuss why we can assume this orthogonality condition.